Formulation en flux des schémas Lagrange-projection et variantes

Introduction à la table ronde

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Outline

Lagrange-remap (LR) schemes – Construction

- Define a piecewise $\mathscr{C}^1$ Lagrange transformation operator $\mathscr{L}(x,t;t_0,x_0)$: $dx/dt = \boldsymbol{u}(x)$, $x(t=0) = x_0$.
- Use the Reynolds transport theorem:

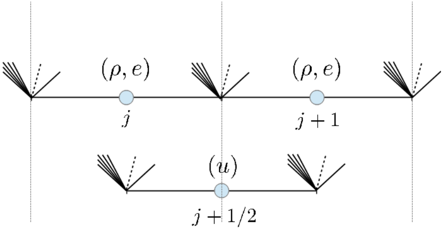
$$\frac{d}{dt}\int_{\Omega_t} q(x,t)\,dx = \int_{\Omega_t} \{\partial_t q + \nabla\cdot(q\boldsymbol{u})\}\,dx$$

  for any moving domain Ω_t, quantity q.
- Finite volume approximation
- Move the mesh according to a Lagrange operator $\mathscr{L}$ over $[t^n, t^{n+1})$
- Project the quantities on the initial mesh.

Attributes of a LR scheme

- Definition of the Lagrange operator $\mathscr{L}$
- Pseudo-viscosity:
  - Linear part: for linear stability;
  - Quadratic part: for $[\![U]\!]^3$ entropy production (nonlinear entropy stability).
- Time integration of the Lagrange step
- Order of accuracy of the projection: requires an interpolator operator $\mathscr{I}$ (Llor, ETSN Roscoff 2012)

Conservative reformulation
(staggered variables, 1D case)



Geometric conservation law (GCL):

$$h_j^{n+1,L} = h + \Delta t\,(u_{j+1/2}^{n+1/2,L} - u_{j-1/2}^{n+1/2,L}).$$

Mass conservation law:

$$\rho_j^{n+1,L} h_j^{n+1,L} = h\rho_j^n.$$

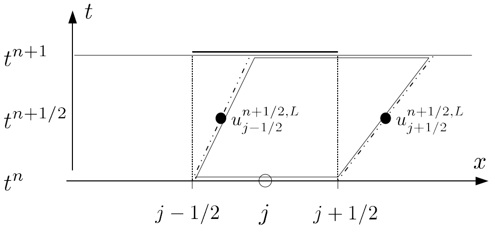
Mass conservation equation

i.e.

$$\rho_j^{n+1,L} = \frac{\rho_j^n}{1+\frac{\Delta t}{h}(\Delta u)_j^{n+1/2,L}}, \quad (\Delta u)_j^{n+1/2,L} := u_{j+1/2}^{n+1/2,L} - u_{j-1/2}^{n+1/2,L}.$$

Projection step:

$$\rho_j^{n+1} = \frac{1}{h}\int_{I_j} \mathscr{I}\rho^{n+1,L}(x)\,dx = \frac{1}{h}\int_{I_j^{n+1,L}} \ldots - \ldots + \ldots\ .$$



Mass equation (cont.)

Mass balance rewritting : under some convenient CFL condition, we have

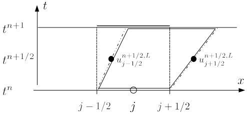
$$\begin{aligned} h\rho_j^{n+1} &= h_j^{n+1,L}\rho_j^{n+1,L} - \Delta t \rho_{j+1/2}^{upw,n+1} u_{j+1/2}^{n+1/2,L} + \Delta t \rho_{j+1/2}^{upw,n+1} u_{j-1/2}^{n+1/2,L} \\ &= h\rho_j^n - \Delta t \rho_{j+1/2}^{upw,n+1} u_{j+1/2}^{n+1/2,L} + \Delta t \rho_{j+1/2}^{upw,n+1} u_{j-1/2}^{n+1/2,L} \end{aligned}$$

in the form

$$\rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{h}\left(\Phi_{m,j+1/2}^{n+1/2,n+1} - \Phi_{m,j-1/2}^{n+1/2,n+1}\right),$$

$$\Phi_{m,j+1/2}^{n+1/2,n+1} = \rho_{j+1/2}^{upw,n+1} u_{j+1/2}^{n+1/2,L}.$$

Flux accuracy



$$\Phi^{n+1/2,n+1}_{m,j+1/2} = \rho^{upw,n+1}_{j+1/2}\, u^{n+1/2,L}_{j+1/2}.$$

Let $\delta x = \dfrac{\Delta t}{2} u(x, t + \Delta t/2)$.

$$\rho(x+\delta x, t+\Delta t)u(x+\delta x, t+\frac{\Delta t}{2}) =$$

$$= \left(\rho(x+\delta x, t+\Delta t/2) + \frac{\Delta t}{2}\partial_t\rho + o(\Delta t)\right) u(x+\delta x, t+\Delta t/2)$$

$$= (\rho u)(x, t+\Delta t/2) + \delta x\,\partial_x(\rho u)(x, t+\Delta t/2)) + \frac{\Delta t}{2}(u\partial_t\rho)(x, t+\Delta t/2) + o(\Delta t)$$

$$= (\rho u)(x, t+\Delta t/2) + \frac{\Delta t}{2} u\left(\partial_t\rho + \partial_x(\rho u)\right)(x, t+\Delta t/2) + o(\Delta t)$$

$$= (\rho u)(x, t+\Delta t/2) + o(\Delta t).$$

Momentum equation (staggered)

$$\frac{\rho_j^{n+1}+\rho_{j+1}^{n+1}}{2}\,u_{j+1/2}^{n+1} = \frac{\rho_j^{n}+\rho_{j+1}^{n}}{2}\,u_{j+1/2}^{n} - \frac{\Delta t}{h}\left(\Phi_{mu,j+1}^{n,n+1} - \Phi_{mu,j}^{n,n+1}\right) \quad (1)$$

with

$$\Phi_{mu,j}^{n,n+1} = \overline{u}_j^{n+1,L}\,\frac{(\overline{\rho}_{j-1/2}^{n+1,L}u_{j-1/2}^{n+1/2,L} + \overline{\rho}_{j+1/2}^{n+1,L}u_{j+1/2}^{n+1/2,L})}{2} + (p+q)_j^{n+1/2,L}. \quad (2)$$

Energy conservation (staggered)

$$\rho_j^{n+1}\widetilde{E}_j^{n+1} = \rho_j^n \widetilde{E}_j^n - \frac{\Delta t}{h}\left(\Phi^{n,n+1}_{mE,j+1/2} - \Phi^{n,n+1}_{mE,j-1/2}\right) \quad (3)$$

with

$$\Phi^{n,n+1}_{mE,j+1/2} = \overline{e}^{n+1,L}_{j+1/2}\, u^{n+1/2,L}_{j+1/2}$$
$$+\frac{1}{2}\left(\frac{1}{2}\overline{(u^2)}^{n+1,L}_j \frac{(\overline{\rho}^{n+1,L}_{j-1/2} u^{n+1/2,L}_{j-1/2} + \overline{\rho}^{n+1,L}_{j+1/2} u^{n+1/2,L}_{j+1/2})}{2}\right.$$
$$\left.+\frac{1}{2}\overline{(u^2)}^{n+1,L}_{j+1} \frac{(\overline{\rho}^{n+1,L}_{j+1/2} u^{n+1/2,L}_{j+1/2} + \overline{\rho}^{n+1,L}_{j+3/2} u^{n+1/2,L}_{j+3/2})}{2}\right) + \Phi^{n+1/2,L}_{pu,j+1/2} \quad (4)$$

Paper submitted

Lagrange-Remap schemes in conservative form

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Presented by Evariste Sanchez-Palenzia

Abstract

Staggered Lagrange-Remap schemes are reformulated as Eulerian finite volume schemes in conservative form, with particular focus on the numerical fluxes. The numerical fluxes naturally involve the states after the Lagrange step. *To cite this article: F. De Vuyst, C. Fochesato et al., C. R. Mecanique 333 (2012).*

Résumé

Ecriture conservative des schémas Lagrange-projection Dans cette Note, on réécrit un schéma Lagrange-projection à variables décalées sous la forme d'un schéma eulérien conservatif classique. Les flux numériques sont naturellement définis à partir des états issus de l'étape Lagrange. *Pour citer cet article : F. De Vuyst, C. Fochesato et al., C. R. Mecanique 333 (2012).*

Key words: Fluid Mechanics ; Euler equations ; Finite Volume ; Lagrange-Remap scheme ; conservative form

Mots-clés : Mécanique des fluides ; Equations d'Euler ; Volumes Finis ; Schémas Lagrange-Projection ; Ecriture conservative

[De Vuyst, Fochesato, Loubère, Rouzier, Saas, Motte, Ghidaglia, CRAS Mécanique, submitted 2012]

Some remarks

Consider $u_{j+1/2}^{n+1/2,L} \geq 0$, then $\rho_{j+1/2}^{upw,n+1} = \rho_j^{n+1,L}$ and

$$\begin{aligned} \Phi_{m,j+1/2}^{n+1/2,n+1} &= \rho_{j+1/2}^{upw,n+1} u_{j+1/2}^{n+1/2,L} \\ &= \frac{\rho_j^n}{1 + \frac{\Delta t}{h}(\Delta u)_j^{n+1/2,L}} u_{j+1/2}^{n+1/2,L}. \end{aligned}$$

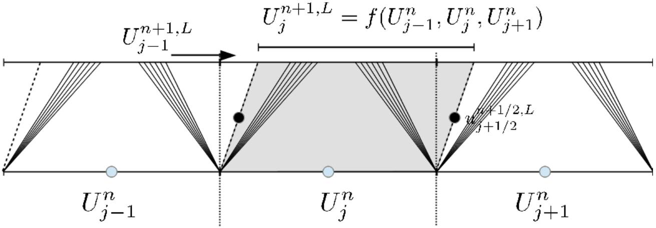
For $t \in (0, \Delta t)$,

$$\Phi_{m,j+1/2}^{n+1/2}(t) = \frac{\rho_j^n}{1 + \frac{t}{h}(\Delta u)_j^{n+1/2,L}} u_{j+1/2}^{n+1/2,L}$$

depends on t, whereas a flux from an approximate Riemann solver does not depend on t (autosimilar solutions of Riemann pbs).

Some remarks (contd)

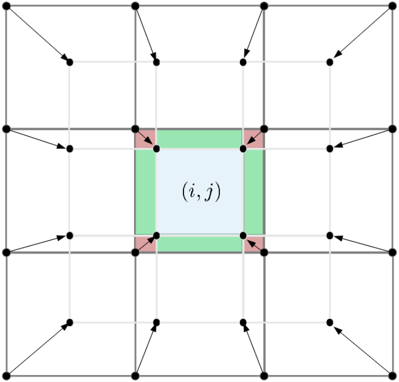
First-order (1st-order remap) LR schemes are actually 5-point schemes !



$$U_j^{n+1} = U_j^n - \frac{\Delta t}{h}\left(\Phi_{j+1/2}^{n,n+1} - \Phi_{j-1/2}^{n,n+1}\right),$$

$$\Phi_{j+1/2}^{n,n+1} = \Phi\left(U_{j-1}^n, U_j^n, U_{j+1}^n, U_{j+2}^n, \Delta t\right).$$

Conservative reformulation (2D case)



⇒ 1st order LR is a 25-point scheme !
Multidimensional corner effects

Conservation reformulation : 2D case

Remap step by Alternating Directions (AD)

See C. Fochesato's talk (this workshop).

Concluding remarks: properties on LR

- Lagrange step + projection: no numerical flux required (except pressure)
- Discrete entropy inequality holds for entropy-satisfying Lagrange steps (under convenient CFL)
- 1D, 1st-order remap: LR is a particular 5-points FV scheme.
- 2D, 1st-order remap : LR is a 25-point scheme (corners) !
- Lagrange step: allows multi-dimensional treatment of linear (contact) waves.
- Projection step can be interpreted as a conservative (flux balance) scheme.
- The underlying numerical flux should not be understood as a flux from a Riemann problem (time-dependent, hidden averages behind projection of the nonlinear waves).

Related works

- Després, Lagoutière, J. Sci. Comp., 2002
- Del Pino, Jourdren, CRAS Série I, 2006
- Kokh-Lagoutière, JCP, 2010

Benefits of a reformulation by flux

- Conservation ensured (up to FP precision)
- Can be integrated in a general FV code
- Parallelism made simple
- Numerical analysis (accuracy, ...)
- Use of FV tools (flux limiters, antidiffusive, ...)

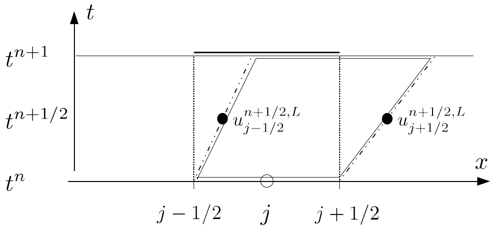
Issue

- Multimaterial case

Toward Lagrange-flux schemes

Design a more general framework of conservative FV eulerian schemes with two ingredients:

- Lagrange solvers (Lagrange step)
- Computation of numerical fluxes



- Optional: (reduced stencil) 3-point 1st-order schemes ?

What about multimaterial in Lagrange-Flux ?

- The question is open;
- There are connections between interface reconstruction, interpolation operator (remap), and antidiffusive fluxes (reservoir [Le Coq, Alouges, De Vuyst, Lorin], [Lagoutière et al.]);
- Flux determination is subject to some requirements: nonlinear diffusion for compactness, second order accuracy, smallest stencil (ref [Llor Roscoff 2012]).
- The secret is certainly behind the choice of the interpolation operator $\mathscr{I}$.
- See also B. Rebourcet's talk (this workshop)